Closing Tue: 12.1, 12.2, 12.3
Closing Thu: 12.4(1), 12.4(2), 12.5(1)
Read my 12.3, 12.4 review sheets. And look at the 12.5 visuals before Wed.

12.4 The Cross Product

We define the cross product, or vector product, for two 3-dimensional vectors,
$\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$ and
$\boldsymbol{b} = \langle b_1, b_2, b_3 \rangle$,
by

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} =$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \boldsymbol{k}$$

$$= (a_2 b_3 - a_3 b_2)\boldsymbol{i} - (a_1 b_3 - a_3 b_1)\boldsymbol{j} + (a_1 b_2 - a_2 b_1)\boldsymbol{k}$$

Ex: $\boldsymbol{a} = \langle 1,2,0 \rangle$ and $\boldsymbol{b} = \langle -1,3,2 \rangle$

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 0 \\ -1 & 3 & 2 \end{vmatrix} =$$

$$(\qquad - \qquad)\boldsymbol{i} - (\qquad - \qquad)\boldsymbol{j} + (\qquad - \qquad)\boldsymbol{k}$$

You do: $\boldsymbol{a} = \langle 1,3,-1 \rangle$, $\boldsymbol{b} = \langle 2,1,5 \rangle$.

Compute $\boldsymbol{a} \times \boldsymbol{b}$

Most important fact:
The vector $\boldsymbol{v} = \mathbf{a} \times \mathbf{b}$ is orthogonal to *both* **a** and **b**.

Note: If **a** and **b** are parallel to each other, then there are many vectors perpendicular to both **a** and **b**.
So what happens to $\boldsymbol{v} = \mathbf{a} \times \mathbf{b}$?

Example: Give me any two vectors that are parallel and let's see.

Right-hand rule

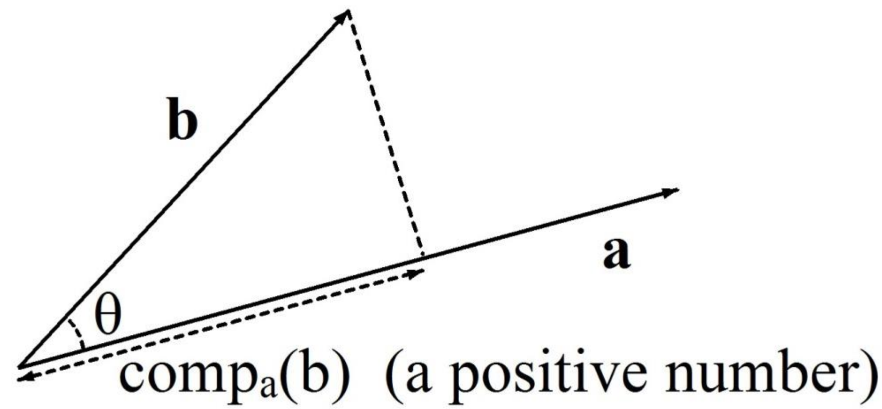
If the fingers of the right-hand curl from **a** to **b**, then the thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.

The magnitude of $\boldsymbol{a} \times \boldsymbol{b}$*:*

Through some algebra and using the dot product rules, it can be shown that

$$|\mathbf{a} \times \mathbf{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin(\theta)$$

where θ is the smallest angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. $(0 \leq \theta \leq \pi)$



Note: $|\mathbf{a} \times \mathbf{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin(\theta)$ is the area of the parallelogram formed by $\boldsymbol{a}$ and $\boldsymbol{b}$

12.5 Intro to Lines in 3D

To describe 3D lines we use parametric equations.

Here is a 2D example

Consider the 2D line: $y = 4x + 5$.

(a) Find a vector parallel to the line. Call it vector **v.**

(b) Find a vector whose head touches some point on the line when drawn from the origin. Call it vector $\mathbf{r_0}$.

(c) We can reach all other points on the line by walking along $\mathbf{r_0}$, then adding scale multiples of **v**.

This same idea works to describe any line in 2- or 3-dimensions.

The equation for a line in 3D:

$\boldsymbol{v} = \langle a, b, c \rangle$ = parallel to the line.

$\boldsymbol{r_0} = \langle x_0, y_0, z_0 \rangle$ = position vector

then all other points, (x, y, z), satisfy

$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + \mathrm{t}\langle a, b, c \rangle$,

for some number t.

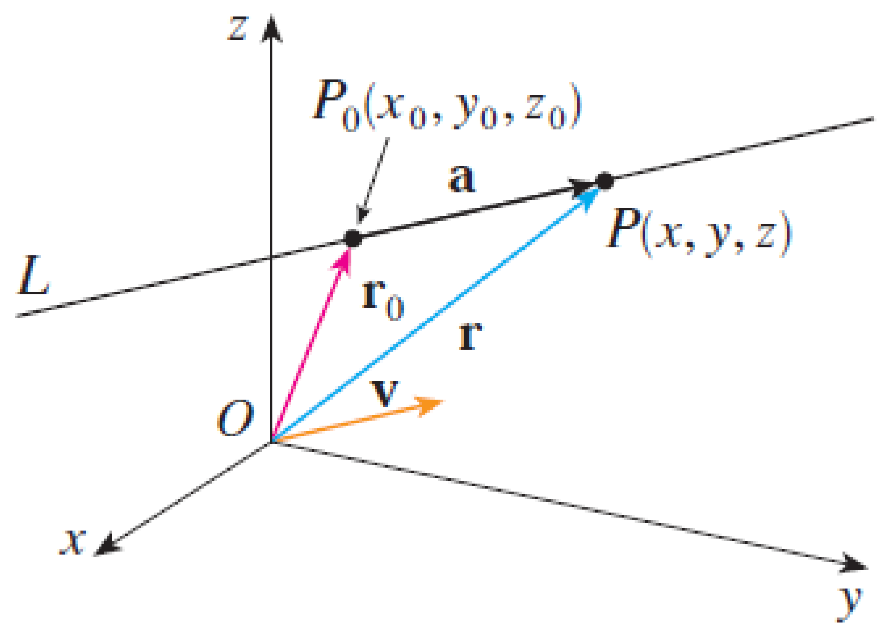
The above form ($\boldsymbol{r} = \boldsymbol{r_0} + \mathrm{t}\,\boldsymbol{v}$) is called the *vector form* of the line.

We also can write this in *parametric form* as:

$$x = x_0 + at,$$
$$y = y_0 + bt,$$
$$z = z_0 + ct.$$

or in *symmetric form*:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$



Basic Example – Given Two Points:
Find parametric equations of the line thru P(3, 0, 2) and Q(-1, 2, 7).

General Line Facts

1. Two lines are **parallel** if their direction vectors are parallel.

2. Two lines **intersect** if they have an (x, y, z) point in common (use different parameters when you combine!)
   Note: The *acute angle of intersection* is the acute angle between the direction vectors.

3. Two lines are **skew** if they don’t intersect and aren’t parallel.